Different Sizes of Infinity

Preamble

In Toy Story, Buzz Lightyear's famous catchphrase is "**To Infinity and Beyond!**". But what does *beyond infinity* mean? Is there anything larger than infinity?
In this poster, we will explore the different "sizes" of infinity, where one infinity can indeed be "larger" than another.
We shall assume familiarity with logic notation and set notation. Remember that infinity is a concept, *not* a number.

Functions

A **function** $f : X \to Y$ is a mapping of *every* element of X to exactly one element of Y; we call X and Y the **domain** and **codomain** of f respectively.
$f : X \to Y$ is **injective** (or *one-to-one*) if each element of Y has at most one element of X that maps to it.

$$\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \implies x_1 = x_2$$

$f : X \to Y$ is **surjective** (or *onto*) if *every* element of Y is mapped to at least one element of X.

$$\forall y \in Y,\ \exists x \in X \text{ s.t. } f(x) = y$$

$f : X \to Y$ is **bijective** (one-to-one correspondence) if it is both injective and surjective: each element of Y is mapped to a unique element of X.

Cardinality and countable sets

X and Y have the same **cardinality**, denoted by $|X| = |Y|$, if there exists a bijection $f : X \to Y$, i.e. two sets have the same size when you can pair each element in one set with a unique element in the other.
The empty set $\varnothing$ is finite and has cardinality $|\varnothing| \coloneqq 0$; a non-empty set X is said to be **finite** and have cardinality $|X| = n \in \mathbb{Z}^+$ if and only if there exists a bijection from X to the set $\{1, 2, \ldots, n\}$.
Generalising this notion, a set X is **countably infinite** if it has the same cardinality as the set $\mathbb{Z}^+$. (This means that to show that X is countably infinite, we need to show that there is a bijection between it and $\mathbb{Z}^+$.)
A set X is **countable** if it is either *finite*, or *countably infinite*; X is **uncountable** if it is not countable.

$2\mathbb{Z}^+$, $\mathbb{Z}$

Is the set of positive even integers larger or smaller than the set of positive integers?
Answer: $|2\mathbb{Z}^+| = |\mathbb{Z}^+|$ as $2\mathbb{Z}^+$ is countably infinite.
Since there exists a bijection $f : \mathbb{Z}^+ \to 2\mathbb{Z}^+$ given by $f(n) = 2n$, hence $2\mathbb{Z}^+$ is countably infinite.

Is the set of integers larger or smaller than the set of positive integers?
Answer: $|\mathbb{Z}| = |\mathbb{Z}^+|$ as $\mathbb{Z}$ is countably infinite.
Since there exists a bijection $f : \mathbb{Z}^+ \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{1-n}{2} & \text{if } n \text{ is odd} \end{cases}$$

where $f(1) = 0$, $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$, $f(6) = 3, \ldots$, i.e. our function f stretches in both positive and negative directions to cover both positive and negative integers, hence $\mathbb{Z}$ is countably infinite.

$\mathbb{R}$

Is the set of real numbers larger or smaller than the set of positive integers?
Answer: $|\mathbb{R}| > |\mathbb{Z}^+|$ as $\mathbb{R}$ is uncountable.
To prove that $\mathbb{R}$ is uncountable, we show that the interval $(0,1)$ is uncountable, which implies the uncountability of $\mathbb{R}$ as $(0,1)$ is a subset of $\mathbb{R}$. Assume otherwise, that $(0,1)$ is countable and there exists a bijection $f : \mathbb{Z}^+ \to (0,1)$:

$$\begin{aligned} f(1) &= 0.a_{1,1}\, a_{1,2}\, a_{1,3}\, a_{1,4}\, a_{1,5} \cdots \\ f(2) &= 0.a_{2,1}\, a_{2,2}\, a_{2,3}\, a_{2,4}\, a_{2,5} \cdots \\ &\ \,\vdots \end{aligned}$$

where $f(i)$ denotes the i-th real number, and each $a_{i,j}$ is some arbitrary non-negative integer.
Define a new real number $b = 0.b_1\, b_2\, b_3\, b_4\, b_5 \cdots$, where $b_i \neq a_{i,i}$ for all $i \in \mathbb{Z}^+$. b differs from every $f(i)$ at least at the i-th decimal place. Hence b is not in our above list, implying f is not surjective and thus not bijective. Thus $(0,1)$ is uncountable, and consequently, $\mathbb{R}$ is uncountable.