Physics Olympiad:
A Guidebook

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Preface

Resources:

- Geeks for Geeks
- Physics Olympiad: Basic to Advanced Exercises
- Kevin Zhou's Physics handouts
- MIT OpenCourseware
- Physics with Elliot

Contents

Part I

Mechanics

1 Newtonian Mechanics

Classical mechanics is all about the motion of **particles**. We start with a definition.

Definition 1.0.1. A **particle** is, loosely, defined as an object of insignificant size.

This means that if you want to say what a particle looks like at a given time, the only information you have to specify is its position.

To describe the position of a particle we need a **reference frame**. This is a choice of origin, together with a set of axes which, for now, we pick to be Cartesian. With respect to this frame, the position of a particle is specified by a vector $\mathbf{x}$. The trajectory of the particle with respect to time is described by

$$\mathbf{x} = \mathbf{x}(t)$$

Notation. In this book we will use both the notation $\mathbf{x}(t)$ and $\mathbf{r}(t)$ to describe the trajectory of a particle.

Definition 1.0.2. The **velocity** of a particle is defined to be

$$\mathbf{v} \equiv \dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} \tag{1.1}$$

Notation. We often denote the time derivative of a variable by a dot above the variable.

Definition 1.0.3. The **acceleration** of the particle is defined to be

$$\mathbf{a} \equiv \ddot{\mathbf{x}} = \frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2} \tag{1.2}$$

Vector Differentiation

The derivative of a vector is defined by differentiating each of the components. For $\mathbf{x} = (x_1, x_2, x_3)$,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \left(\frac{\mathrm{d}x_1}{\mathrm{d}t}, \frac{\mathrm{d}x_2}{\mathrm{d}t}, \frac{\mathrm{d}x_3}{\mathrm{d}t}\right)$$

Geometrically, the derivative of a path $\mathbf{x}(t)$ lies tangent to the path.

We will also be working with vector differential equations. These should be viewed as three, coupled differential equations – one for each component. We will frequently come across situations where we need to differentiate vector dot-products and cross-products. The meaning of these is easy to see if we use the chain rule on each component. For example, given two vector functions of time, $\mathbf{f}(t)$ and $\mathbf{g}(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f}\cdot\mathbf{g}) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}\cdot\mathbf{g} + \mathbf{f}\cdot\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f}\times\mathbf{g}) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}\times\mathbf{g} + \mathbf{f}\times\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}$$

Note that the order that we write the dot product does not matter, but we have to be more careful with the cross product because, for example,

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}\times\mathbf{g} = -\mathbf{g}\times\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}.$$

§1.1 Newton's Laws

Newtonian mechanics is a framework which allows us to determine the trajectory $\mathbf{x}(t)$ of a particle in any given situation. This framework is usually presented as three axioms known as **Newton's laws of motion**. They are given by:

Theorem 1.1.1 (Newton's 1st Law of Motion)**.** Left alone, a particle moves with constant velocity.

Theorem 1.1.2 (Newton's 2nd Law of Motion)**.** The acceleration (or, more precisely, the rate of change of momentum) of a particle is proportional to the force acting upon it.

Theorem 1.1.3 (Newton's 3rd Law of Motion)**.** Every action has an equal and opposite reaction.

§1.2 Inertial Frames and Newton's First Law

We have introduced the idea of a frame of reference: a Cartesian coordinate system in which you measure the position of the particle. But for reference frames such as rotating ones, for a particle of trajectory $\mathbf{x}(t)$, we certainly won't find that $d^2\mathbf{x}/dt^2 = 0$, i.e. particles do not travel at constant velocity.

We see that if we want Newton's first law to hold, we must be more careful about the kind of reference frames we're talking about. We first define an inertial reference frame.

Definition 1.2.1. An **inertial reference frame** is one in which particles travel at constant velocity when the force acting on it vanishes.

In other words, in an inertial frame,

$$\ddot{\mathbf{x}} = 0 \quad \text{when} \quad \mathbf{F} = 0$$

The true content of Newton's first law can then be better stated as: inertial frames exist.

§1.2.1 Galilean Relativity

Inertial frames are not unique. Given an inertial frame S in which a particle has coordinates $\mathbf{x}(t)$, we can always construct another inertial frame S' in which the particle has coordinates $\mathbf{x}'(t)$ by any combination of the following transformations:

- Translation: $\mathbf{x}' = \mathbf{x} + \mathbf{a}$, for constant $\mathbf{a}$.
- Rotation: $\mathbf{x}' = \mathbf{R}\mathbf{x}$, for a 3×3 matrix $\mathbf{R}$ obeying $\mathbf{R}^T\mathbf{R} = 1$. (This also allows for reflections if $\det \mathbf{R} = -1$, although our interest will primarily be on continuous transformations).
- Boost: $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$, for constant velocity $\mathbf{v}$.

Remark. The three transformations above are not quite the unique transformations that map between inertial frames. But, for most purposes, they are the only interesting ones! The others are transformations of the form $\mathbf{x}' = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$. This is just a trivial rescaling of the coordinates; for example, we can measure distances in S in units of metres and distances in S' in units of parsecs.

It is simple to prove that all of these transformations map one inertial frame to another. Suppose that a particle moves with constant velocity with respect to frame S, so that $d^2\mathbf{x}/dt^2 = 0$. Then, for each of the transformations above, we also have $d^2\mathbf{x}'/dt^2 = 0$ which tells us that the particle also moves at constant velocity in S'. Or, in other words, if S is an inertial frame then so too is S'. The three transformations generate a group known as the *Galilean group*.

We have already mentioned that Newton's second law is to be formulated in an *inertial frame*. But, importantly, it doesn't matter which inertial frame. In fact, this is true for all laws of physics: they are the same in any inertial frame. This is known as the **principle of relativity**.

So position, direction and velocity are relative. But acceleration is not. You do not have to accelerate relative to something else. It makes perfect sense to simply say that you are accelerating or you are not accelerating. In fact, this brings us back to Newton's first law: if you are not accelerating, you are sitting in an inertial frame.

§1.2.2 Absolute Time

There is one last issue that we have left implicit in the discussion above: the choice of time coordinate t. If observers in two inertial frames S and S' fix the units – seconds, minutes, hours – in which to measure the duration time then the only remaining choice they can make is when to start the clock. In other words, the time variable in S and S' differ only by

$$t' = t + t_0$$

This is sometimes included among the transformations that make up the Galilean group.

The existence of a uniform time, measured equally in all inertial reference frames, is referred to as **absolute time**. It is something that we will have to revisit when we discuss special relativity. As with the other Galilean transformations, the ability to shift the origin of time is reflected in an important property of the laws of physics. The fundamental laws don't care when you start the clock. All evidence suggests that the laws of physics are the same today as they were yesterday. They are **time translationally invariant**.

§1.3 Newton's Second Law

The second law is the meat of the Newtonian framework. It is the famous "$F = ma$", which tells us how a particle's motion is affected when subjected to a force $\mathbf{F}$. The correct form of the second law is

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\mathbf{x}}) = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) \tag{1.3}$$

This is usually referred to as the equation of motion. The quantity in brackets is called the **momentum**:

$$\mathbf{p} \equiv m\dot{\mathbf{x}}$$

where m is the (inertial) mass of the particle.

In cases where mass does not change with time, we can write the second law in the more familiar form:

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) \tag{1.4}$$

Newton's equation is a second order differential equation; this means that we will have a unique solution only if we specify two initial conditions. These are usually taken to be the position $\mathbf{x}(t_0)$ and the velocity $\dot{\mathbf{x}}(t_0)$ at some initial time t_0.

2 Kinematics

§2.1 Uniformly accelerated linear motion

§2.1.1 Equations of motion

Velocity as function of time:

$$v_x(t) = v_{x0} + a_x t \tag{2.1}$$

Derivation. For the one-dimensional case in the x-direction, from the definition of acceleration as the time derivative of velocity,

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t}$$

Solving the differential equation,

$$\int_{v_{x0}}^{v_x(t)} \mathrm{d}v_x = \int_0^t a_x \, \mathrm{d}t \implies v_x(t) - v_{x0} = a_x t$$

$\therefore \mathbf{v}(t) = (v_{x0} + a_x t)\hat{\imath} + (v_{y0} + a_y t)\hat{\jmath}$ □

Displacement as function of time:

$$x(t) = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \tag{2.2}$$

Derivation.

$$\begin{aligned}
v_x(t) &= \frac{\mathrm{d}x(t)}{\mathrm{d}t} \\
v_{x0} + a_x t &= \frac{\mathrm{d}x(t)}{\mathrm{d}t} \\
\mathrm{d}x &= (v_{x0} + a_x t)\,\mathrm{d}t \\
\int_{x_0}^{x(t)} &= \int_0^t (v_{x0} + a_x t)\,\mathrm{d}t \\
x(t) - x_0 &= v_{x0}t + \frac{1}{2}a_x t^2 \\
x(t) &= x_0 + v_{x0}t + \frac{1}{2}a_x t^2
\end{aligned}$$

$\therefore \mathbf{r}(t) = (x_0 + v_{x0}t + \frac{1}{2}a_x t^2)\hat{i} + (y_0 + v_{y0}t + \frac{1}{2}a_y t^2)\hat{j}$ □

Eliminating time dependence:

$$v_x(x)^2 = {v_{x0}}^2 + 2a_x[x(t) - x_0] \tag{2.3}$$

Derivation.

$$\begin{aligned} a_x(t) &= \frac{\mathrm{d}v_x(t)}{\mathrm{d}t} \\ a_x &= \frac{\mathrm{d}v_x}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ a_x &= v_x \frac{\mathrm{d}v_x}{\mathrm{d}x} \\ a_x \,\mathrm{d}x &= v_x \,\mathrm{d}v_x \\ \int_{x_0}^{x(t)} a_x \,\mathrm{d}x &= \int_{v_{x0}}^{v_x(t)} v_x \,\mathrm{d}v_x \\ \frac{1}{2}v_x(t)^2 - \frac{1}{2}{v_{x0}}^2 &= a_x[x(t) - x_0] \\ v_x(x)^2 &= {v_{x0}}^2 + 2a_x[x(t) - x_0] \end{aligned}$$

□

§2.1.2 Projectile motion

Horizontal and vertical motions are completely *independent* from each other.

Conventionally, $+x$-direction is horizontally rightward, $+y$-direction is vertically upward.

Horizontal motion	**Vertical motion**
$v_x(t) = v_{x0} + a_x t$	$v_y(t) = v_{y0} + a_y t$
$x(t) = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$	$y(t) = y_0 + v_{y0}t + \frac{1}{2}a_y t^2$
$v_x(x)^2 = {v_{x0}}^2 + 2a_x(x - x_0)$	$v_y(y)^2 = {v_{y0}}^2 + 2a_y(y - y_0)$

The trajectory of two dimensional free falling motion is given by

$$\begin{aligned} x(t) &= x_0 + v_0 \cos\theta t \implies t = \frac{x(t) - x_0}{v_0 \cos\theta} \\ y(t) &= y_0 + v_0 \sin\theta t - \frac{1}{2}gt^2 \\ &= y_0 + \tan\theta[x(t) - x_0] - \left(\frac{g}{2{v_0}^2 \cos^2\theta}\right)[x(t) - x_0]^2 \end{aligned}$$

Hence, the trajectory is *parabolic*.

Exercise 2.1.1

The acceleration of a marble in a certain fluid is proportional to the speed of the marble squared and is given by $a = -kv^2$. If the marble enters the fluid with a speed of v_0, how long will it take before the marble's speed is half of its initial value?

Solution. Rewriting acceleration as the derivative of velocity and solving the differential equation,

$$a(t) = -kv(t)^2 \implies \frac{\mathrm{d}v}{\mathrm{d}t} = -kv^2$$

Solving the differential equation,

$$\frac{1}{v^2}\,\mathrm{d}v = -k\,\mathrm{d}t \implies \int_{v_0}^{\frac{v_0}{2}} \frac{1}{v^2}\,\mathrm{d}v = -\int_0^t k\,\mathrm{d}t \implies \frac{2}{v_0} - \frac{1}{v_0} = kt \implies \boxed{t = \frac{1}{kv_0}}$$

To determine the displacement of the marble at this time, rewrite acceleration as time derivative of velocity.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -kv^2$$

Using chain rule,

$$\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = -kv^2$$

Since $v = \frac{\mathrm{d}x}{\mathrm{d}t}$,

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -kv^2$$

Solving the differential equation,

$$\frac{1}{v}\,\mathrm{d}v = -k\,\mathrm{d}x \implies \int_{v_0}^{\frac{v_0}{2}} \frac{1}{v}\,\mathrm{d}v = -\int_0^x k\,\mathrm{d}x \implies \ln\frac{v_0}{2} - \ln v_0 = -kx \implies \boxed{x = \frac{\ln 2}{k}}$$

□

Exercise 2.1.2

Ship A is 10km due west of ship B. Ship A is heading directly north at a speed of 30km/h while ship B is heading in a direction 60° west of north at a speed of 20km/h. What will be their distance of closest approach?

Solution. We first set up a coordinate system: choose origin at initial position of ship A, $+x$-direction is eastward and $+y$-direction is northward.

Position vector of A with respect to B:

$$\begin{aligned} \mathbf{r}_{AB} &= \mathbf{r}_{AG} + \mathbf{r}_{GB} \\ &= \mathbf{r}_{AG} - \mathbf{r}_{BG} \\ &= v_A t\hat{j} + (10 - v_B t\sin 60°)\hat{i} + v_B t\cos 60°\hat{j} \\ &= (-10 + v_B \sin 60° t)\hat{i} + (v_A t - v_B \cos 60° t)\hat{j} \end{aligned}$$

Relative distance between A and B at time t:

$$r_{AB} = |\mathbf{r}_{AB}| = \sqrt{(-10 + v_B \sin 60° t)^2 + (v_A t - v_B \cos 60° t)^2}$$

To find minimum value of r_{AB},

$$\frac{\mathrm{d}r_{AB}}{\mathrm{d}t} = 0 \implies t_0 = \frac{\sqrt{3}}{7}$$

$\therefore$ Minimum distance between A and B = $\boxed{7.56\,\text{km}}$. □

Exercise 2.1.3

A projectile is fired up an incline of angle ϕ with an initial speed v_i at an angle θ with respect to the horizontal ($\theta > \phi$). Find the direction in which it should be aimed to achieve the maximum range along the incline. What is the maximum range?

Solution. $\mathbf{v}_x(t) = v_i \cos\theta, \mathbf{v}_y(t) = v_i \cos\theta$ Let the time when projectile lands on the incline be T.

□

Exercise 2.1.4

At $t = 0$ on a planet, a projectile is fired with speed v_0 at an angle θ above the horizontal. On this planet, the acceleration due to gravity increases linearly with time, starting with a value of zero when the projectile is fired from the ground, i.e. $g(t) = \alpha t$. What horizontal distance does the projectile travel? What should θ be to maximise this distance?

3 Translational Dynamics

§3.1 Forces

§3.1.1 Types of forces

Weight

Weight $\mathbf{W}$ is the gravitational force exerted by Earth on an object.

Normal force

Normal force $\mathbf{N}$ is the contact force exerted by a surface (ground or floor) on an object.

Tension force

Tension force $\mathbf{T}$ is the force experienced in an object when it is deformed (compressed or depressed).

Spring force

Theorem 3.1.1: Hooke's Law

Spring force is directly proportional to extension of spring.

$$\mathbf{F}_s = -k\mathbf{x} \tag{3.1}$$

where k is the spring constant.

Frictional force

There are two types of frictional force:

- **Kinetic friction force**: object is sliding on rough surface

$$f_k = \mu_k N$$

- **Static friction force**: object is not sliding

$$f_s \leq \mu_s N$$

Resistive force

Drag force F_R: force caused by interaction of an object and the fluid it is moving through

- For objects moving at low speeds: resistive force is directly proportional to speed

$$\mathbf{F}_R = -b\mathbf{v} \tag{3.2}$$

- For objects moving at high speeds: resistive force is directly proportional to square of speed

$$\mathbf{F}_R = -\frac{1}{2}D\rho A\mathbf{v}^2 \tag{3.3}$$

  where D is the drag coefficient, which depends on the shape and surface texture of the object.

  Terminal velocity is when an object moving through a fluid has reached translational equilibrium. For an object falling downwards:

$$\sum \mathbf{F}_y = \mathbf{F}_g - \mathbf{F}_R = m\mathbf{a}_y \implies mg - \frac{D}{\rho}Av^2 = ma_y \implies a_y = g - \frac{D\rho Av^2}{2m}$$

  In absence of air resistance, $\mathbf{a}_y = g$.

  When $\mathbf{a}_y = 0$,

$$\frac{D\rho Av^2}{2m} = g \implies \boxed{v_{\text{terminal}} = \sqrt{\frac{2mg}{D\rho A}}}$$

§3.2 Centre of mass

Definition 3.2.1. The **centre of mass** is a special point in a system, as if all of the mass of the system is concentrated at that point.

Centre of mass for a **system of point particles**:

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} \quad y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i} \quad z_{CM} = \frac{\sum_i m_i z_i}{\sum_i m_i} \tag{3.4}$$

where the distances depend on the *coordinate system* set up.

Centre of mass of an **extended object** (think of an extended object as a system containing infinitely many small mass elements):

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{1}{M} \sum_i x_i \Delta m_i$$

$$x_{CM} = \frac{1}{M} \int x \, \mathrm{d}m \quad y_{CM} = \frac{1}{M} \int y \, \mathrm{d}m \quad z_{CM} = \frac{1}{M} \int z \, \mathrm{d}m \tag{3.5}$$

§3.2.1 Motion of centre of mass

Velocity

x-, y- and z-components of the velocity of centre of mass, denoted by $v_{CM,x}$, $v_{CM,y}$ and $v_{CM,z}$, are the time derivatives of x_{CM}, y_{CM} and z_{CM} respectively.

$$v_{CM,x} = \frac{\sum_i m_i v_{xi}}{\sum_i m_i} \quad v_{CM,y} = \frac{\sum_i m_i v_{yi}}{\sum_i m_i} \quad v_{CM,z} = \frac{\sum_i m_i v_{zi}}{\sum_i m_i} \tag{3.6}$$

These equations can be written as one single vector equation:

$$\mathbf{v}_{CM} = \frac{1}{M} \sum_i m_i \mathbf{v}_i \tag{3.7}$$

Total momentum of the system is given by

$$\mathbf{p} = M\mathbf{v}_{CM} = \sum_i m_i \mathbf{v}_i \tag{3.8}$$

This equation states that the total momentum is the product of total mass and velocity of centre of mass.

Acceleration and external force

Taking time derivative of the above equation gives

$$M\mathbf{a}_{CM} = \sum_i m_i \mathbf{a}_i$$

Note that $\sum_i m_i\mathbf{a}_i$ is simply the sum of all forces (external and internal):

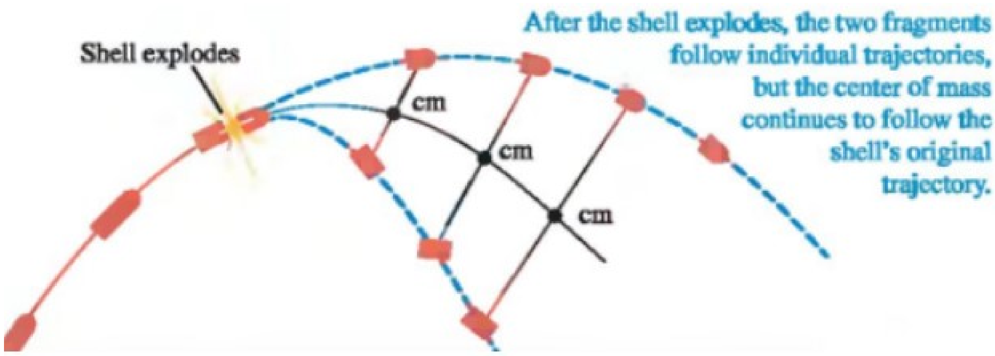
$$\sum \mathbf{F} = \sum \mathbf{F}_{ext} + \sum \mathbf{F}_{int} = \sum_i m_i\mathbf{a}_i$$

By Newton's 3rd law, internal forces all cancel in pairs so $\sum \mathbf{F}_{int} = 0$. Hence

$$\sum \mathbf{F}_{ext} = M\mathbf{a}_{CM} \quad \text{and} \quad \sum \mathbf{F}_{ext} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{3.9}$$

When a body or a collection of particles is acted on by external forces, centre of mass moves as though all the mass were concentrated at that point and it were acted on by a net force equal to the sum of external forces on the system.

For example, a shell explodes into two fragments in flight. Ignoring air resistance, centre of mass continues on the same trajectory aas the shell's path before exploding.



Note that if the net external force acting on the system is zero, we get

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0 \implies M\mathbf{v}_{CM} = \mathbf{p} = \text{constant}$$

§3.3 Equilibrium

Equilibrium conditions:

1. force balance (vectorially or in terms of projections)

2. torque balance (only for one- and two-dimensional geometry).

Stable and unstable equilibria.

§3.4 Elastic modulus

compression. We consider three types of deformation and define an elastic modulus for each:

1. **Young's modulus** measures the resistance of a solid to a change in its length.
2. **Shear modulus** measures the resistance to motion of the planes within a solid parallel to each other.
3. **Bulk modulus** measures the resistance of solids or liquids to changes in their volume.

§3.4.1 Tensile and compressive

We usually assume objects to be rigid. When large forces are applied to an object, it deforms.

Suppose that we pull on the ends of a bar with a force F. We say that the bar is in tension. The internal forces in the bar resist the tension forces and hold the bar together. Even so, the bar deforms and the equilibrium length of the bar increases.

If the bar is in equilibrium with the applied forces, then every cross section of the bar must be subject to the same internal forces that resist stretching.

Definition 3.4.1. **Stress** is the ratio of the magnitude of the applied force F to cross-sectional area A.

$$\sigma \equiv \frac{F}{A} \tag{3.10}$$

Remark. There are two types of stress: tensile and compressive.

Definition 3.4.2. **Strain** is the ratio of the change in length δ to the initial length L.

$$\varepsilon \equiv \frac{\delta}{L} \tag{3.11}$$

Remark. There are two types of strain: tensile and compressive.

The amount of strain an object undergoes depends on the stress applied to it. If the stress is not too great, the strain is observed to be proportional to the stress.

Definition 3.4.3. **Young modulus** is the ratio of stress to strain.

$$Y \equiv \frac{\sigma}{\varepsilon} = \frac{FL}{A\delta} \tag{3.12}$$

§3.4.2 Shear

When an external force acts on an object, it undergoes deformation. If the direction of the force is parallel to the plane of the object. The deformation will be along that plane. The stress experienced by the object here is shear stress.

Definition 3.4.4. **Shear stress** is a type of stress that acts coplanar with cross section of material.

$$\tau \equiv \frac{F}{A} \tag{3.13}$$

Shear stress arises due to shear forces. They are the pair of forces acting on opposite sides of a body with the same magnitude and opposite direction.

§3.5 Work Done and Energy

§3.5.1 Work

Work done by a constant force:

$$W = \mathbf{F} \cdot \Delta\mathbf{r} = F\Delta r \cos\theta \tag{3.14}$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{r}$.

Work done by a non-constant force:

$$W = \int_{x_i}^{x_f} F_x \, \mathrm{d}x \tag{3.15}$$

§3.5.2 Energy

Theorem 3.5.1: Net Work–Kinetic Energy Theorem

$$\sum W = \Delta K \tag{3.16}$$

Kinetic energy for translational motion:

$$K = \frac{1}{2}mv^2 \tag{3.17}$$

Gravitational potential energy (in constant gravitational field):

$$U_g = mgh \tag{3.18}$$

Potential energy for simple force fields (also as a line integral of the force field).

Relationship between conservative forces and potential energy:

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x} \tag{3.19}$$

§3.5.3 Power

Definition 3.5.1: Power

Rate at which work is done

$$P_{\text{avg}} = \frac{W}{\Delta t} \tag{3.20}$$

$$P_{\text{instantaneous}} = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{3.21}$$

Instantaneous power (constant force):

$$P_{\text{instantaneous}} = \mathbf{F} \cdot \mathbf{v} \tag{3.22}$$

Derivation.

$$P_{\text{instantaneous}} = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{F} \cdot \Delta\mathbf{r})}{\mathrm{d}t} = F \cdot \frac{\mathrm{d}\Delta\mathbf{r}}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{v}$$

□

4 Rotational Motion

§4.1 Kinematics

Definition 4.1.1. **Radian** is defined as

$$\theta \equiv \frac{s}{r} \tag{4.1}$$

Definition 4.1.2. **Angular displacement** of a rigid object is the angle that the object rotates through during some time interval.

$$\Delta\theta \equiv \theta_f - \theta_i \tag{4.2}$$

Remark. Every point on a rigid object undergoes the same angular displacement in any given time interval.

Definition 4.1.3. Angular velocity is the rate of change of angular displacement with respect to time.

$$\omega(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \tag{4.3}$$

Remark. Every part of a rotating rigid object has the same angular velocity at any instant of time.

Direction of angular velocity can be found using the "right hand rule". Curl fingers of right hand around rotation. Thumb points in the direction of the vector.

Definition 4.1.4. **Angular acceleration** is the rate of change of angular velocity with respect to time.

$$\alpha(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t} \tag{4.4}$$

Angular velocity as a function of time

$$\omega(t) = \omega_0 + \alpha t \tag{4.5}$$

Angular position as a function of time

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{4.6}$$

Eliminating time dependence

$$\omega^2(t) = {\omega_0}^2 + 2\alpha[\theta(t) - \theta_0] \tag{4.7}$$

For constant a and α, we can write analogous equations for rotational motion as in linear motion, as shown above.

Tangential velocity (linear speed):

$$v = r\omega \tag{4.8}$$

Derivation.

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}(r\theta)}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = r\omega$$

□

Tangential acceleration:

$$a_t = r\alpha \tag{4.9}$$

Derivation.

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}(r\omega)}{\mathrm{d}t} = r\frac{\mathrm{d}\omega}{\mathrm{d}t} = r\alpha$$

□

Centripetal acceleration

$$a_r = \frac{v^2}{r} = r\omega^2 \tag{4.10}$$

Derivation.

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}$$
$$\Delta v = \frac{v}{r}\Delta s$$
$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v}{r} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

□

Acceleration is the vector sum of tangential acceleration and centripetal acceleration.

$$\mathbf{a} = \mathbf{a_t} + \mathbf{a_r}$$
$$|\mathbf{a}| = \sqrt{{a_t}^2 + {a_r}^2} = r\sqrt{\alpha^2 + \omega^4} \tag{4.11}$$

Comparison between translational and rotational motion:

Quantity	Translational	Rotational
Displacement	x	θ
Velocity	v	ω
Acceleration	a	α
Mass / moment of inertia	m	I
Momentum	p	L

§4.2 Dynamics

§4.2.1 Moment of Inertia

Definition 4.2.1. **Moment of inertia** is the measure of the resistance of an object to changes in its rotational motion, depends on the *choice of rotational axis.*

Moment of inertia of one particle:

$$I \equiv mr^2 \tag{4.12}$$

Moment of inertia of a system of particles:

$$I = \sum_i m_i {r_i}^2 \tag{4.13}$$

Moment of inertia of a continuous rigid object (divide it into infinitely many small elements):

$$I = \lim_{\Delta m_i \to 0} \sum_i {r_i}^2 \Delta m_i$$

$$I = \int r^2 \, \mathrm{d}m \tag{4.14}$$

Moments of inertia of homogeneous rigid objects:

Object	Moment of inertia
Hoop about central axis	$I = MR^2$
Solid cylinder (or disk) about central axis	$I = \frac{1}{2}MR^2$
Solid cylinder (or disk) about central diameter	$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
Thin rod about axis through center perp to length	$I = \frac{1}{12}ML^2$
Solid sphere about any diameter	$I = \frac{2}{5}MR^2$
Thin spherical shell about diameter	$I = \frac{2}{3}MR^2$
Hoop about any diameter	$I = \frac{1}{2}MR^2$

Expressions for **mass densities** come in useful:

$$\mathrm{d}m = \begin{cases} \lambda \, \mathrm{d}x & \text{linear mass density} \\ \sigma \, \mathrm{d}x & \text{surface mass density} \\ \rho \, \mathrm{d}x & \text{volume mass density} \end{cases}$$

Exercise 4.2.1

Moment of inertia of a uniform thin hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.

Solution. For constant radius, moment of inertia is given by

$$I = \int r^2 \, \mathrm{d}m = R^2 \int \mathrm{d}m = \boxed{MR^2}$$

□

Exercise 4.2.2

Moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its centre of mass.

Solution. Set up coordinate system: rod lies along x-axis, axis lies along y-axis.

A small length $\mathrm{d}x$ at a distance x from origin has a mass $\mathrm{d}m$. Let λ be **linear mass density**, then

$$\lambda = \frac{M}{L} = \frac{\mathrm{d}m}{\mathrm{d}x} \implies \mathrm{d}m = \frac{M}{L} \, \mathrm{d}x$$

Moment of inertia is given by

$$I = \int r^2 \, \mathrm{d}m = \int x^2 \frac{M}{L} \, \mathrm{d}x = \frac{M}{L} \int_{\frac{L}{2}}^{\frac{L}{2}} x^2 \, \mathrm{d}x = \frac{M}{L} \left[\frac{2}{3} \left(\frac{L}{2} \right)^3 \right] = \boxed{\frac{1}{12} ML^2}$$

□

Exercise 4.2.3

Moment of inertia of uniform solid cylinder has a radius R, mass M and length L about its axis of cylinder.

Solution. The solid cylinder has to be cut or split into infinitesimally thin rings. Each ring consists of the thickness of $\mathrm{d}r$ with length L. We then sum up the moments of these infinitesimally thin cylindrical shells.

Using the concept of volume mass density ρ,

$$\mathrm{d}m = \rho \, \mathrm{d}V = \rho(L \, \mathrm{d}A) = \rho L(\pi(r + \mathrm{d}r)^2 - \pi r^2) = (2\pi r)L\rho \, \mathrm{d}r$$

Moment of inertia is given by

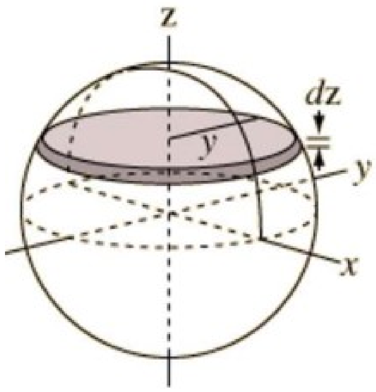
$$I = \int r^2 \, \mathrm{d}m = \int_0^R 2\pi r^3 L\rho \, \mathrm{d}r = 2\pi L\rho \int_0^R r^3 \, \mathrm{d}r = \frac{1}{2}(\pi r^2 L\rho)R^2 = \boxed{\frac{1}{2} MR^2}$$

□

Exercise 4.2.4

Moment of inertia of a solid sphere of mass M and radius R about an axis through its centre.

Solution. The expression for the moment of inertia of a sphere can be developed by summing the moment of infinitely thin disks about the z-axis through its centre.



Using volume mass density:

$$\rho = \frac{M}{V} = \frac{\mathrm{d}m}{\mathrm{d}V} \implies \mathrm{d}m = \rho\,\mathrm{d}V$$

Moment of inertia of one disk about z-axis:

$$\mathrm{d}I = \frac{1}{2}y^2\,\mathrm{d}m = \frac{1}{2}y^2\rho\,\mathrm{d}V = \frac{1}{2}y^2\rho\pi y^2\,\mathrm{d}z$$

Hence moment of inertia of sphere about z-axis is given by

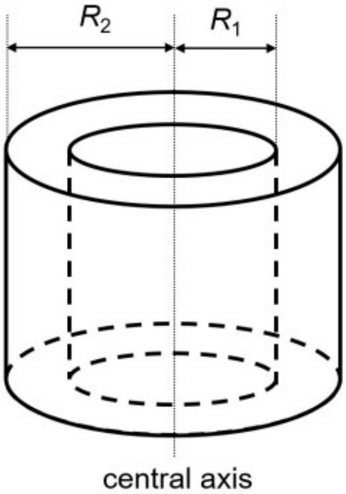
$$I_{CM} = \int \mathrm{d}I = \frac{1}{2}\rho\pi\int_{-R}^{R} y^4\,\mathrm{d}z = \frac{1}{2}\rho\pi\int_{-R}^{R}(R^2 - z^2)^2\,\mathrm{d}z = \frac{8}{15}\rho\pi R^5 = \boxed{\frac{2}{5}MR^2}$$

Remark. Another method is to sum up spherical hollow shells.

□

Exercise 4.2.5

Moment of inertia of a hollow cylinder with inner radius R_1 and outer radius R_2 about the central axis.



Proof. Consider a hollow cylindrical shell with radius r and height L.

Using volume mass density,

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V} \implies \mathrm{d}m = \rho\,\mathrm{d}V = 2\pi\rho L r\,\mathrm{d}r$$

Hence moment of inertia is

$$I = \int r^2\,\mathrm{d}m = \int_{R_{\text{in}}}^{R_{\text{out}}} 2\pi\rho L r^3\,\mathrm{d}r = \frac{\pi\rho L}{2}\left[r^4\right]_{R_{\text{in}}}^{R_{\text{out}}} = \boxed{\frac{1}{2}M(R_{\text{out}}^2 + R_{\text{in}}^2)}$$

where $\rho = \dfrac{M}{\pi(R_{\text{out}}^2 - R_{\text{in}}^2)L}$. □

Theorem 4.2.1: Parallel axis theorem

Moment of inertia I about any axis parallel to axis through CM and a distance D away is given by

$$I = I_{CM} + MD^2 \tag{4.15}$$

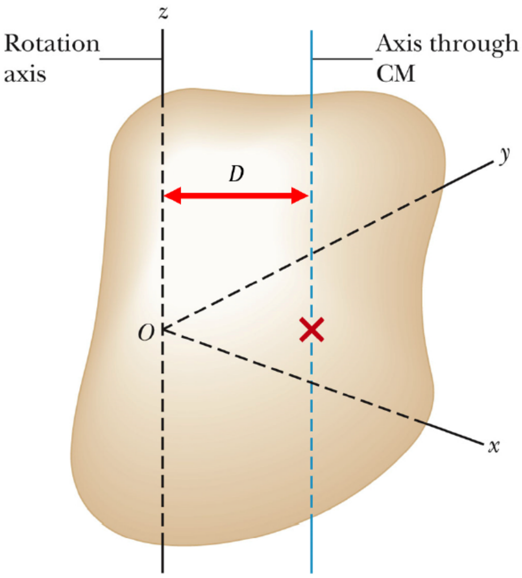


Figure 4.1: Parallel axis theorem

Proof. Moment of inertia about the z-axis is

$$I = \int r^2 \, dm = \int (x^2 + y^2) \, dm$$

From the figure, $x = x' + x_{CM}$ and $y = y' + y_{CM}$ hence

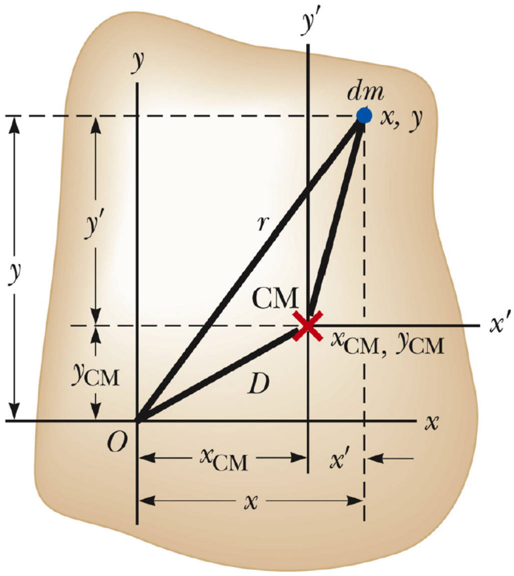
$$\begin{aligned} I &= \int \left[(x' + x_{CM})^2 + (y' + y_{CM})^2 \right] dm \\ &= \int (x'^2 + y'^2) \, dm + ({x_{CM}}^2 + {y_{CM}}^2) \int dm + 2x_{CM} \int x' \, dm + 2y_{CM} \int y' \, dm \end{aligned}$$

By definition of centre of mass, $\int x' \, dm = \int y' \, dm = 0$.

Given that $\int dm = M$ and $D^2 = {x_{CM}}^2 + {y_{CM}}^2$,

$$\boxed{I = I_C M + MD^2}$$

□



Exercise 4.2.6

Consider a uniform rigid rod of mass M and length L. Find the moment of inertia of the rod about an axis perpendicular to the rod through one end.

Solution. Moment of inertia is

$$\frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \boxed{\frac{1}{3}ML^2}$$

□

Theorem 4.2.2: Perpendicular axis theorem

Sum of moments of inertia about any two perpendicular axes in the plane of the body is equal to the moment of inertia about an axis through the point of intersection, perpendicular to the plane of the object.

$$I_z = I_x + I_y \tag{4.16}$$

This theorem works only for planar figures (2D bodies), i.e. bodies of negligible thickness.

Proof.

$$\begin{aligned} I_z &= \int r^2 \, dm \\ &= \int (x^2 + y^2) \, dm \\ &= \int x^2 \, dm + \int y^2 \, dm \end{aligned}$$

□

§4.2.2 Rotational kinetic energy

We treat a rigid object as a collection of particles rotating about a fixed z-axis with an angular speed ω.

Kinetic energy of the i-th particle is given by

$$K_i = \frac{1}{2} m_i {v_i}^2 = \frac{1}{2} m_i (r_i \omega)^2 = \frac{1}{2} m_i {r_i}^2 \omega^2$$

Hence rotational kinetic energy possessed by a rigid object is given by

$$K = \sum_i K_i = \frac{1}{2} \left(\sum_i m_i {r_i}^2 \right) \omega^2$$

$$K = \frac{1}{2} I \omega^2 \tag{4.17}$$

Remark. Notice the similarities between this and the formula for translational kinetic energy $K = \frac{1}{2} m v^2$.

§4.2.3 Torque

Definition 4.2.2. **Torque** is the measure of tendency of a force to *rotate* an object about some axis.

$$|\tau| = F\ell = F \cdot r\sin\theta \tag{4.18}$$

where $\ell = r\sin\theta$ is the **level arm**, i.e. the perpendicular distance from axis of rotation to the line of action of force.

Representing torque as a vector,

$$\tau = \mathbf{r} \times \mathbf{F} \tag{4.19}$$

Torque and angular acceleration

$$\tau = I\alpha \tag{4.20}$$

Proof. Consider a force $\mathrm{d}\mathbf{F}_t$ acting on a mass element $\mathrm{d}m$ of an extended object.

From Newton's 2nd law,

$$\mathrm{d}\mathbf{F}_t = (\mathrm{d}m)a_t \implies d\tau = r\,\mathrm{d}\mathbf{F}_t = ra_t\,\mathrm{d}m$$

Since $a_t = r\alpha$,

$$\mathrm{d}\tau = \alpha r^2\,\mathrm{d}m$$

Net torque about origin due to all external forces is

$$\sum \tau = \int \mathrm{d}\tau = \int \alpha r^2\,\mathrm{d}m = \alpha \int r^2\,\mathrm{d}m = \boxed{I\alpha}$$

□

§4.2.4 Work, Energy, Power

The work done by force F on an object as it rotates through an infinitesimal distance $\mathrm{d}s = r\,\mathrm{d}\theta$ is

$$\mathrm{d}W = F\,\mathrm{d}s = (F\sin\phi)r\,\mathrm{d}\theta = \tau\,\mathrm{d}\theta$$

Integrating both sides gives

$$W = \int \tau\,\mathrm{d}\theta \tag{4.21}$$

Rate at which work is done by F as the object rotates about the fixed axis through an angle $\mathrm{d}\theta$ in a time interval $\mathrm{d}t$ is

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \tau\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Hence power is

$$P = \tau\omega \tag{4.22}$$

Theorem 4.2.3: Work–kinetic energy theorem

Net work done by external forces in rotating a rigid body about a fixed axis equals the change in the object's rotational kinetic energy.

$$\sum W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 \tag{4.23}$$

§4.2.5 Rolling motion

In pure rolling motion, an object rolls without slipping.

The object rotates through an angle θ, so centre of mass moves a linear distance $s = R\theta$.

Linear speed of centre of mass:

$$v_{CM} = R\omega \tag{4.24}$$

Derivation.

$$v_{CM} = \frac{\mathrm{d}s}{\mathrm{d}t} = R\frac{\mathrm{d}\theta}{\mathrm{d}t} = R\omega$$

□

Linear acceleration of centre of mass:

$$a_{CM} = R\alpha \tag{4.25}$$

Derivation.

$$a_{CM} = \frac{\mathrm{d}v_{CM}}{\mathrm{d}t} = R\frac{\mathrm{d}\omega}{\mathrm{d}t} = R\alpha$$

□

Pure rolling motion is a combination of

- pure **translational motion** of centre of mass
- pure **rotational motion** around centre of mass

Total kinetic energy of a rolling object is the **sum** of rotational kinetic energy about centre of mass and translational kinetic energy of centre of mass.

$$K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}M{v_{CM}}^2 \tag{4.26}$$

Derivation. Object rotates about point P, the point of contact with ground.

Total kinetic energy can be expressed as

$$K = \frac{1}{2}I_P\omega^2$$

where I_P is the moment of inertia about an axis through P.

Using parallel axis theorem,

$$I_P = I_{CM} + MR^2$$

Hence

$$K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}M(R\omega)^2 = \boxed{\frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{CM}^2}$$

□

§4.2.6 Angular momentum

Definition 4.2.3. Angular momentum is the cross product of instantaneous position vector $\mathbf{r}$ and linear momentum $\mathbf{p}$.

$$L \equiv \mathbf{r} \times \mathbf{p} \tag{4.27}$$

Derivation. From the definition of torque, $\tau \equiv \mathbf{r} \times \mathbf{F}$,

$$\sum \tau = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

Add the term $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p}$ to the right-hand side, which is zero because $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}$ which is parallel to $\mathbf{p}$.

$$\sum \tau = \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p} = \frac{\mathrm{d}(\mathbf{r} \times \mathbf{p})}{\mathrm{d}t} = \boxed{\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}}$$

□

Hence the torque acting on a particle is equal to the time rate of change of angular momentum.

$$\sum \tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \tag{4.28}$$

This is the rotational analog of Newton's 2nd law.

System of particles

Total angular momentum of a system of particles about some point is defined as the vector sum of angular momentum of the individual particles.

$$L_{total} = \sum_i L_i$$

Hence

$$\sum_i \tau_i = \sum_i \frac{\mathrm{d}L_i}{\mathrm{d}t} = \frac{\mathrm{d}L_{total}}{\mathrm{d}t}$$

The torque acting on the particles of the system are due to internal forces between particles and external forces. However, net torque due to internal forces is zero as a result of Newton's 3rd law. Hence total angular momentum of system varies only if net external torque acts on the system:

$$\sum \tau_{\text{ext}} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \tag{4.29}$$

Net torque about axis through origin equals time rate of change of total angular momentum of system about that origin.

Theorem 4.2.4

Resultant torque acting on a system about axis through centre of mass equals time rate of change of angular momentum regardless of motion of centre of mass.

Rigid body

Angular momentum of one particle is

$$L \equiv mr^2\omega = mrv \tag{4.30}$$

Taking the sum of angular momentum over all particles on a rigid body,

$$\mathbf{L} = \sum_i \mathbf{L}_i = \left(\sum_i m_i {r_i}^2\right)\omega = I\omega$$
$$L = I\omega \tag{4.31}$$

Taking time derivative,

$$\frac{\mathrm{d}L}{\mathrm{d}t} = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\alpha$$

Hence

$$\sum \tau_{ext} = I\alpha \tag{4.32}$$

Conservation of angular momentum

Theorem 4.2.5

Total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.

$$\sum \tau_{ext} = \frac{\mathrm{d}L_{total}}{\mathrm{d}t} = 0 \tag{4.33}$$
$$L_i = L_f$$

Exercise 4.2.7

A uniform disc of radius R is spinning about the vertical axis and placed on a horizontal surface. If the initial angular speed is ω and the coefficient of friction is μ, determine the time before which the disc comes to rest.

Solution. A common mistake is to directly apply the equation $\tau = f_k \times R$ because the radius is not the same for all points on the disc.

Instead, we analyse using a ring of mass $\mathrm{d}m$, radius r and width $\mathrm{d}r$, where all points on the ring have the same radius from the centre.

Torque of ring is

$$\mathrm{d}\tau = r\,\mathrm{d}f_k = r(\mu_k g\,\mathrm{d}m) = \mu_k g r\,\mathrm{d}m$$

Torque of disk is

$$\tau = \int \mathrm{d}\tau = \int \mu_k g r\,\mathrm{d}m = \mu_k g \int r\,\mathrm{d}m$$

Using **area density**,

$$\sigma = \frac{M}{A} = \frac{\mathrm{d}m}{\mathrm{d}A},$$

hence

$$\frac{\mathrm{d}m}{2\pi r} = \frac{M}{\pi R^2} \implies \mathrm{d}m = \frac{M}{\pi R^2} 2\pi r$$

Substituting this gives us

$$\tau = \mu_k g \int r \frac{M}{\pi R^2} 2\pi r\,\mathrm{d}r = \frac{2\mu_k M g}{R^2} \int_0^R r^2\,\mathrm{d}r = \frac{2\mu_k M g}{R^2}\frac{R^3}{3} = \boxed{\frac{2}{3}\mu_k M g R}$$

Using Newton's 2nd Law,

$$\tau = I\alpha \implies \frac{2}{3}\mu_k M g R = \left(\frac{1}{2} M R^2\right)\alpha \implies \alpha = \frac{4}{3}\frac{\mu_k g}{R}$$

Using angular acceleration, calculate the time taken:

$$\begin{aligned}
\omega_f &= \omega_i - \alpha t \\
0 &= \omega - \alpha t \\
t &= \frac{\omega}{\alpha} \\
\Aboxed{t &= \frac{3}{4}\frac{R\omega}{\mu_k g}}
\end{aligned}$$

□

Exercise 4.2.8

A uniform rod of mass M and length L is placed vertically with one end pinned to a frictionless horizontal floor. It starts to fall when it is given a small displacement. When the rod makes an angle θ with the vertical, find

(a) the radial acceleration of the top of the rod;

(b) the tangential acceleration of the top of the rod.

Solution. Let τ_O denote torque about pin at point O.

Using Newton's 2nd Law,

$$\tau_O = I\alpha \implies Mg\left(\frac{L}{2}\sin\theta\right) = \left[\frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2\right]\alpha \implies \alpha = \frac{3}{2}\frac{g\sin\theta}{L}$$

Note that since the axis of rotation through the pin at O is parallel to the axis through centre of mass, we use the *parallel axis theorem* to determine I.

Using this value of angular acceleration, we can calculate tangential acceleration.

$$a_t = L\alpha = \boxed{\frac{3}{2}g\sin\theta}$$

To calculate radial acceleration, recall that

$$a_r = \frac{{v_t}^2}{L} = L\omega^2$$

To find ω, recall that angular acceleration is the derivative of angular velocity with respect to time.

$$\begin{aligned}
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{3}{2}\frac{g\sin\theta}{L} \\
\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{3}{2}\frac{g\sin\theta}{L} \\
\omega\,\mathrm{d}x &= \frac{3}{2}\frac{g\sin\theta}{L}\,\mathrm{d}\theta \\
\int_0^\omega \omega\,\mathrm{d}\omega &= \int_0^\theta \frac{3}{2}\frac{g\sin\theta}{L}\,\mathrm{d}\theta \\
\frac{\omega^2}{2} &= \frac{3g}{2L}(-\cos\theta + 1) \\
\omega^2 &= \frac{3g}{L}(1-\cos\theta)
\end{aligned}$$

Substituting this value of angular velocity gives us

$$\boxed{a_r = 3g(1-\cos\theta)}$$

□

5 Gravitation

§5.1 Gravitational Force

§5.1.1 Newton's Law of Gravitation

Theorem 5.1.1: Newton's Law of Gravitation

Let the masses be m_1 and m_2 and $\mathbf{r}$ be their separation. Then the gravitational force acting on each due to the presence of the other is given by

$$\mathbf{F} = G\frac{m_1 m_2}{|\mathbf{r}|^2}\hat{\mathbf{r}} \tag{5.1}$$

where G is the gravitational constant which has the value of 6.67×10^{11} N m^2 kg^{-2}

§5.1.2 Principle of Superposition

Given a group of n particles, we find the net (or resultant) gravitational force on any one of them from the others by using the principle of superposition, by adding individual gravitational forces vectorially:

$$\mathbf{F}_{k,\text{net}} = \sum_{1 \le i \le n, i \ne k} \mathbf{F}_{ki}$$

We have looked at the case of point charges. What about the gravitational force on a particle from an extended object? This force is found by dividing the object into parts small enough to treat as particles and then finding vector sum of the forces on the particle from all the parts. In the limiting case, we can divide the extended object into differential parts each of mass dm and each producing a differential force d$\mathbf{F}$ on the particle. In this limit, the sum becomes an integral and we have

$$\mathbf{F}_k = \int \mathrm{d}\mathbf{F}\,.$$

§5.2 Gravitational Field

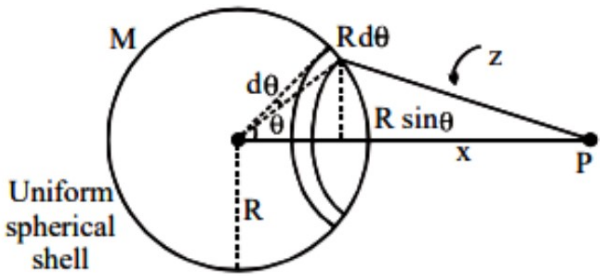
§5.2.1 Gravitational Near Earth's Surface

§5.2.2 Gravitation Inside Earth

§5.3 Gravitational Potential Energy

§5.3.1 Gravitational potential energy of spherical shell

Let us consider a uniform spherical shell of mass M. Its mass per unit area will be $\sigma = \frac{M}{4\pi R^2}$. Consider a strip of width $R\,\mathrm{d}\theta$ and radius $R\sin\theta$. The whole shell is made up of such strips.



Potential at P due to the ring is given by

$$\mathrm{d}V = -G\frac{\mathrm{d}M}{z}$$

$$\mathrm{d}V = -G\frac{2\pi R^2 \sin\theta\,\mathrm{d}\theta}{z} \tag{1}$$

Now,

$$z = \sqrt{(x - R\cos\theta)^2 + (R\sin\theta)^2}$$
$$z^2 = x^2 + R^2 - 2xR\cos\theta$$

Differentiating, we get

$$2xR\sin\theta = 2z\,\mathrm{d}z$$

$$\sin\theta\,\mathrm{d}\theta = \frac{z\,\mathrm{d}z}{xR} \tag{2}$$

From (1) and (2),

$$\mathrm{d}V = -\frac{GM}{2z}\frac{z\,\mathrm{d}z}{xR} = -\frac{GM}{2xR}\mathrm{d}z$$

Case 1: P lies outside the shell

In this case $x - R \leq z \leq x - R$. Therefore potential is

$$V = -\frac{GM}{2xR}\int_{x-R}^{x+R}\mathrm{d}z = -\frac{GM}{2xR}2R = -\frac{GM}{x}$$

Case 2: P lies inside the shell

In this case $R - x \leq z \leq x - R$. Therefore potential is

$$V = -\frac{GM}{2x} \int_{R-x}^{x+R} \mathrm{d}z = -\frac{GM}{R}$$

§5.3.2 Elliptical orbits and orbital transfers

To solve problems involving orbital transfers, the key strategy is to work from energy considerations in satellite motion. Recall that the total mechanical energy E of a bound satellite system is $E = -\frac{GMm}{2r}$.

A similar expression for E for elliptical orbits is the same, with r replaced by the semi-major axis of length a:

$$E = -\frac{GMm}{2a}$$

§5.3.3 Effective radial potential

An orbiting satellite of mass m under the influence of the gravitational field due to the Earth of mass M, is at a distance r from the centre of Earth.

Assuming that the system consists of Earth and a satellite and the mass of Earth is many times larger than that of satellite, total energy U of the system is given by

$$E_{\text{total}} = \frac{1}{2}mv_r^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

where L is angular momentum of satellite, v_r is radial velocity of satellite.

Derivation. Total energy of a moving satellite m under the influence of the gravitational field due to the Earth of mass M is given by:

$$U = E_k + E_p = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Since the satellite has an ellipsoidal orbit,

$$v^2 = v_t^2 + v_r^2$$

Since satellite is in a central force field $\tau = \mathbf{r} \times \mathbf{F} = \mathbf{0}$ and $L = mrv_t$.

Therefore, sub into equations and simplifying,

$$E_{\text{total}} = \frac{1}{2}mv_t^2 + \frac{1}{2}mv_r^2 - \frac{GMm}{r} = \boxed{\frac{1}{2}mv_r^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}}$$

□

Hence **effective potential** is given by

$$U_{\text{eff}} = E_{\text{total}} - \text{KE}_r = \boxed{\frac{L^2}{2mr^2} - \frac{GMm}{r}}$$

$$U_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{5.2}$$

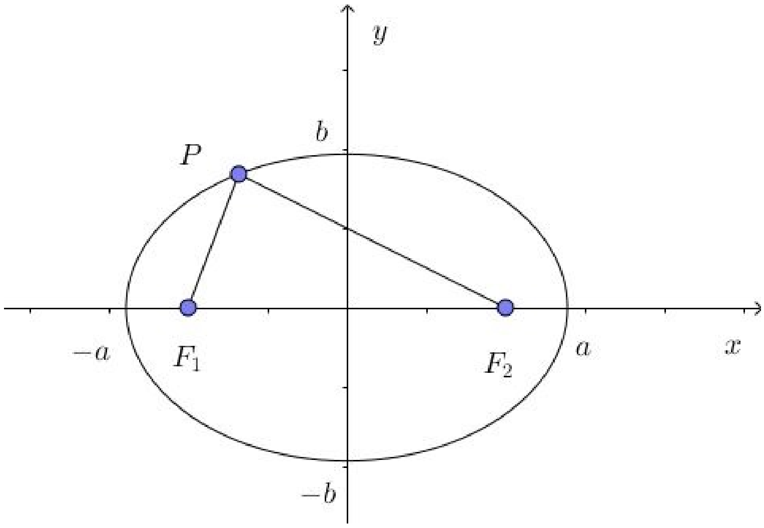
§5.4 Planets and Satellites: Kepler's Laws

Kepler's Laws are used to describe planetary motion.

Theorem 5.4.1: Kepler's 1st Law

The orbit of every planet is an ellipse with the Sun at one of the two foci.

Proof. Kepler's 1st law indicates that the circular orbit is a very special case and elliptical orbits are the general situation.



An ellipse is mathematically defined by choosing two points F_1 and F_2, each of which is called a focus, and then drawing a curve through points for which $PF_1 + PF_2$ is constant.

The major axis is the longest distance through the centre between points on the ellipse. Semi-major axis is the distance a. The minor axis is the shortest distance. Semi-minor axis is the distance b. Eithr focus of the ellipse is located at a distance c from the centre of ellipse, where $a^2 = b^2 + c^2$.

The eccentricity of an ellipse is defined as $e = \frac{c}{a}$, which describes the general shape of the ellipse. For a circle, $c = 0$. Higher values of eccentricity correspond to longer and thinner ellipses. The range of values of the eccentricity for an ellipse is $0 < e < 1$.

Aphelion: point where the planet is the farthest away from the Sun (for object in orbit around Earth, this point is called the **apogee**), distance $= a + c$

Perihelion: point where the planet is the nearest to the Sun (for object in orbit around Earth, this point is called the **perigee**), distance $= a - c$ □

Theorem 5.4.2: Kepler's 2nd Law

A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Proof. Kepler's 2nd law is a consequence of the conservation of angular momentum.

By the law of conservation of angular momentum,

$$L = rp = mr^2\omega = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Area of a small sector $\mathrm{d}A$ swept out by the radial line is given by

$$\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{2}\frac{L}{m}\,\mathrm{d}t \implies A = \frac{1}{2}\frac{Lt}{m}$$

Hence for constant t, A is constant. □

Theorem 5.4.3: Kepler's 3rd Law

The ratio of the square of a planet's period of revolution to the cube of the semi-major axis of its orbit around the Sun is a constant, and this constant is the same for all planets.

$$T^2 = \frac{4\pi^2}{GM} a^3 \tag{5.3}$$

where a is the length of the semi-major axis.

Proof. In the case of a circular orbit, gravitational force provides centripetal force for orbit.

$$mr\omega^2 = G\frac{mM}{r^2}$$

Then, expressing the angular velocity ω in terms of the orbital period T and then rearranging, results in Kepler's Third Law:

$$mr\left(\frac{2\pi}{T}\right)^2 = G\frac{Mm}{r^2} \implies T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \implies T^2 \propto r^3$$

□

§5.5 Satellites: Orbits and Energy

§5.6 Einstein and Gravitation

6 Hydrodynamics

§6.1 Fluid Statics

Theorem 6.1.1: Pascal's principle

In equilibrium, the pressure in a static varies with height:

$$\frac{\mathrm{d}P}{\mathrm{d}y} = -\rho g \tag{6.1}$$

This always holds in equilibrium. For instance, if we squeeze a sealed container of fluid, increasing the pressure locally, then this pressure increase must propagate throughout the entire fluid to maintain $dP/dy = -\rho g$.

Theorem 6.1.2: Archimedes' Principle

An object in a fluid experiences an upward buoyant force due to the different pressures on its top and bottom sides. The force is equal in magnitude to the weight of the fluid displaced.

§6.1.1 Surface tension

Surface tension and the associated energy, capillary pressure.

§6.2 Fluid Mechanics

Steady flow is where every fluid particle arriving at a given point has the same velocity.

Viscosity is the degree of internal friction; resistance that 2 adjacent layers of the fluid have to move relative to each other.

Ideal fluid flow:

1. Non-viscous
2. Steady
3. Incompressible
4. Irrotational

Theorem 6.2.1: Equation of continuity

This equation says that the flow of fluid through a tube of changing cross section is constant.

$$A_1 v_1 = A_2 v_2 \tag{6.2}$$

Derivation. Conservation of mass

Consider the case of a fluid moving from a region of cross-sectional area A_1 to a region of area A_2. Since the fluid is incompressible, the same amount of it leaves each region and enters the other region during the same time interval.

Volume of fluid that flows into the tube across A_1 in time interval Δt is

$$\Delta V_1 = A_1 v_1 \Delta t$$

Hence the mass of fluid that flows into the tube in time Δt is

$$\Delta m_1 = \rho \Delta V_1 = \rho A_1 v_1 \Delta t$$

Similarly, the mass of fluid that flows across A_2 is

$$\Delta m_2 = \rho \Delta V_2 = \rho A_2 v_2 \Delta t$$

Equating the two masses,

$$\Delta m_1 = \Delta m_2 \implies \boxed{A_1 v_1 = A_2 v_2}$$

□

Theorem 6.2.2: Bernoulli's equation

$$P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g y_2 = \text{constant} \tag{6.3}$$

Derivation. Conservation of energy

□

§6.2.1 Viscosity

Theorem 6.2.3: Poiseuille's Law

$$P_1 - P_2 = 8\frac{Q\eta L}{\pi R^4} \tag{6.4}$$

where Q is the flow rate, η is the coefficient of viscosity.